PHYSICAL REVIEW VOLUME 135, NUMBER 5B 7 SEPTEMBER 1964

Calculation of the Proton-Neutron Mass Difference by S-Matrix Methods*

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(Received 27 April 1964)

The proton-neutron mass difference is calculated using S-matrix methods developed in the previous paper. Neutrons and protons are treated as bound-state poles in the π-N scattering amplitude, and the mass difference is obtained by finding the electromagnetic corrections to their binding energies. The results are in good agreement with experiment. No cutoffs or other purely theoretical parameters are involved. All the long-range electromagnetic corrections to the π-N interaction are investigated. Photon exchange turns out to be the most important. Form factors appear as short-range modifications of the photon exchange force. The results of the calculation are not sensitive to the detailed behavior of the form factors at large momentum transfer.

I. INTRODUCTION

It is generally believed that the proton-neutron mass difference is electromagnetic in origin. If this is correct, one should, in principle, be able to calculate $\delta M = M_p - M_n$ by adding electromagnetism to a charge-independent set of strong-interaction equations. In practice, one is, of course, forced to make approximations.

Feynman and Speisman[1] were the first to point out that the observed mass difference $\delta M_{\rm exp} \approx -1.3$ MeV is not inconsistent with the known charges and magnetic moments of the nucleons. Their method consisted of integrating the ordinary self-energy diagrams with cutoffs which could be interpreted as form factors or a breakdown of quantum electrodynamics. Subsequently, this method has been shown[2] to be equivalent to keeping only the lowest mass intermediate state ($N\gamma$) in a presumably exact dispersion relation, and a number of authors[3–5] have attempted to calculate the mass difference by combining the Feynman-Speisman approach with the experimental form factors. Unfortunately, the self-energy integrals are sensitive to the high-momentum transfer behavior of the form factors, and no one has been able to obtain the observed mass difference without introducing an undetermined "core" parameter. Since this method for calculating δM seems to require a knowledge of the form factors for momentum transfers on the order of a few $(\text{BeV})^2$, one begins to wonder if it is a good approximation to ignore the higher intermediate states, such as $N\pi\gamma$. In short, the "self-energy" approach has shown that δM could be of electromagnetic origin, but has not produced reliable quantitative results.

More recent developments in dispersion theory have led to the hypothesis that all strongly interacting particles are composite.[6] In particular, nucleons are supposed to be bound states containing components of $\pi+N$, $\rho+N$, $K+\Sigma$, and other systems with the correct quantum numbers. From this point of view, one should be able to calculate the proton-neutron mass difference by finding the electromagnetic corrections to their binding energies. The aim of this paper is to show that this approach does, in fact, lead to a quantitative explanation of the mass difference.

We will consider nucleons as bound states appearing in the πN scattering amplitude. The choice of the πN rather than, say, the $K\Sigma$ amplitude is dictated by the practical considerations that (i) the πN system is the least massive two-particle system with the proper quantum numbers; (ii) the nucleon is strongly coupled to it (i.e., the πN component of the nucleon wave function is certainly large); and (iii) we have a fairly good understanding of the pion-nucleon interaction. Using a technique developed in the previous paper,[7] hereafter referred to as (I), we can then calculate the nucleon electromagnetic mass difference. Our theoretical estimate for the mass difference turns out to be $\delta M_{\rm theo} \approx -1.4$ MeV. In carrying out the calculation, one does not encounter any small difference between large numbers. No cutoffs or other purely theoretical parameters are involved. Barring the unlikely possibility that this agreement between theory and experiment is entirely accidental, this result would seem to confirm the usual assumption that the strong part of the π-N interaction conserves isospin exactly, and all deviations from charge independence are of electromagnetic origin.

From our point of view, the proton-neutron mass difference is the result of a difference in the forces which act in the two-charge states of the $J=\frac{1}{2}^+$, $T=\frac{1}{2}$, πN system. Since only the $T_3=-\frac{1}{2}$ state contains two charged particles (i.e., $\pi^- p$), photon exchange can be

* Work supported in part by the U. S. Atomic Energy Commission. The work reported here is included in a thesis to be submitted by Roger F. Dashen to the California Institute of Technology in partial fulfillment of the requirements for the degree of Doctor of Philosophy.
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[1] R. Feynman and G. Speisman, Phys. Rev. **94**, 500 (1954).
[2] M. Cini, E. Ferrari, and R. Gatto, Phys. Rev. Letters **2**, 7 (1959).
[3] S. Sunakawa and K. Tanaka, Phys. Rev. **115**, 754 (1959).
[4] H. Katsumori and M. Shimada, Phys. Rev. **124**, 1203 (1961).
[5] A. Solomon, Nuovo Cimento **27**, 748 (1963).
[6] G. Chew and S. Frautschi, Phys. Rev. Letters **8**, 41 (1962). For a treatment of the nucleon as a bound state in the πN system, see E. Abers and C. Zemach, Phys. Rev. **131**, 1205 (1963).
[7] R. Dashen and S. Frautschi, Phys. Rev. **135**, B1190 (1964). Referred to in the text as (I).

expected to create a significant difference in the forces. In fact, it turns out that photon exchange is responsible for the bulk of the p-n mass difference. The electromagnetic form factors of the pions and nucleons will modify the photon exchange force at small distances. This will naturally affect our estimate for δM. However, convergence can be maintained without the form factors and our results are not particularly sensitive to the high-momentum transfer behavior of the form factors.

In the following section, the dispersion integrals for δM are written down and the general nature of the input singularities is discussed. It turns out that an approximate evaluation of the dispersion relation can lead to a spurious infrared divergence. A method for handling these divergences is outlined in Sec. III. For reasons given in (I), it is believed that when the spurious infrared divergences are treated in this particular manner, the dispersion relation will be dominated by low mass singularities. In Sec. IV we carry out the detailed task of estimating δM by keeping only the nearby singularities.

This paper is not meant to be self-contained. The basic equations and scheme for handling the infrared divergence problem were derived in (I). For a discussion of π-N scattering in the absence of electromagnetic effects, the reader is referred to papers by Frazer and Fulco,[8] and Frautschi and Walecka.[9]

II. FORMAL CONSIDERATIONS

Let us begin by supposing that the electromagnetic interactions have been turned off. Conservation of isospin will then be exact and the nucleons will have a common mass M and the pions a mass μ. We are interested in the nucleon pole which occurs in the analytically continued π-N scattering amplitude. This pole appears in the $J=\frac{1}{2}$, $L=1$, $T=\frac{1}{2}$ channel for which we define the amplitude

$$\begin{aligned} A(W)&=\rho(W)e^{i\eta}\sin\eta\,,\\ \rho(W)&=\frac{W^2}{M^2}\frac{1}{(W-M)^2-\mu^2}\frac{1}{q}\,,\\ \eta&=\text{phase shift}\,,\\ W&=\text{total c.m. energy}\,,\\ q&=\text{c.m. momentum}\,, \end{aligned} \tag{1}$$

and the inelasticity factor

$$I(W)=\tfrac{1}{2}\rho(W)\,|e^{2i\eta}|\,. \tag{2}$$

With the above choice for ρ, A has no kinematic singularities in the W plane[8,9] and for $|W-M|\ll M$, $A(W)\approx e^{i\eta}\sin\eta/q^3$. The residue of the nucleon pole in the direct channel turns out to be $-3f^2/\mu^2$; $f^2\approx 0.08$.

[8] W. Frazer and J. Fulco, Phys. Rev. **119**, 1420 (1960).
[9] S. Frautschi and J. Walecka, Phys. Rev. **120**, 1486 (1960).

We now turn on the electromagnetic interactions and thereby destroy the previous equality of the $T_3=\pm\frac{1}{2}$ amplitudes. In particular, the $T_3=+\frac{1}{2}$ amplitude now has a pole at the mass of the proton while the $T_3=-\frac{1}{2}$ amplitude has a pole at the mass of the neutron. Since the masses of the nucleons and pions are no longer degenerate, the kinematics become more complicated and the kinematic factors ρ will be different for the two charge states. Since we only are interested in the difference between the proton and neutron masses, it is convenient to define

$$\begin{aligned} \delta M&=M_p-M_n\,,\\ \delta A&=A(+\tfrac{1}{2})-A(-\tfrac{1}{2})\,,\\ \delta I&=I(+\tfrac{1}{2})-I(-\tfrac{1}{2})\,,\\ \delta\rho&=\rho(+\tfrac{1}{2})-\rho(-\tfrac{1}{2})\,, \end{aligned} \tag{3}$$

where the indices $\pm\frac{1}{2}$ refer to states of $T=\frac{1}{2}$ and $T_3=\pm\frac{1}{2}$. [Note that the definitions of δA, δI, and $\delta\rho$ are slightly different from those used in (I).]

The perturbation techniques developed in (I) can now be used to calculate δM to order $\alpha\approx 1/137$. We suppose that the unperturbed amplitude has been obtained in the form N/D with D normalized such that $D'(M)=1$. The analog of Eq. (22) in (I) then becomes

$$\delta M=-\frac{\mu^2}{3f^2}\Bigg[\frac{1}{2\pi i}\int_{\circlearrowright L}\frac{\delta A(W')D^2(W')}{W'-M}dW'-\frac{1}{\pi}\times\int_{M+\mu}^{\infty}\frac{|D(W')|^2\delta I(W')-\frac{1}{2}\,\mathrm{Re}(D^2(W')\delta\rho(W'))}{W'-M'}dW'\Bigg], \tag{4}$$

where the contour L encloses all the singularities of δA which lie to the left of $\mathrm{Re}W=M+\mu$. The term containing $\delta\rho$ is a kinematic correction [cf. Eq. (24) in (I)].

It is convenient to separate the singularities of δA into two classes. The first type of singularity is of purely kinematic origin. The position of a singularity in the W plane and the kinematic factors which affect its strength are functions of the masses of the scattered particles. When the nucleon and pion mass splittings are taken into account, the original strong interaction singularities in $A(+\frac{1}{2})$ and $A(-\frac{1}{2})$ will have slightly different positions and strengths, simply because the kinematics are different in the two channels. The more distant singularities will be affected very little but there will be an imperfect cancellation between the lower mass part of the original singularities in $A(+\frac{1}{2})$ and $A(-\frac{1}{2})$. The second type of singularity in δA comes from corrections to the unitarity condition for the T matrix, $\mathrm{Im}T_{ab}\propto\sum_c T_{ac}T_{bc}{}^*$. These will appear either because a new intermediate state has become available (e.g., $N\bar{N}\rightarrow\gamma\rightarrow 2\pi$ in the t channel), or because of mass shifts in an already existing intermediate state, or because of electromagnetic corrections in a vertex or amplitude leading to one of the original intermediate states (e.g., electromagnetic corrections to π-N scattering in the u channel).

To order α, any electromagnetic correction must transform under rotations in isospin space like I, T_3, or T_3^2, where I is the unit operator and T_3 is the third component of isotopic spin. For nucleons, $T_3^2 = I/4$, and only those corrections which transform like T_3 will contribute to δM. Two important consequences of this observation are (i) since the pion mass differences transform like T_3^2, they cannot affect δM; (ii) a photon which appears in an intermediate state c of the unitarity condition $\mathrm{Im}T_{ab} \propto \sum_c T_{ac}T_{cb}^*$ will not contribute to δM unless it connects one isovector vertex and one isoscalar vertex. The fact that many of the electromagnetic corrections to the πN interaction do not affect δM makes the calculation of the mass difference surprisingly simple. A calculation of the individual nucleon mass shifts would be a more formidable task.

III. TREATMENT OF SPURIOUS INFRARED DIVERGENCES

In (I) it was pointed out that, in principle, Eq. (4) is convergent in the limit of vanishing photon mass, but in an approximate calculation a spurious infrared divergence will probably appear. A simple prescription given in (I) for removing a spurious divergence runs as follows. One writes the infrared divergent part of $\delta\eta$ in the form $\delta\eta_{\mathrm{infrared}} = f(W)\ln(\lambda/g(W))$, where λ is the photon mass. The factor $f(W)$ is uniquely determined and can be calculated in perturbation theory. The function $g(W)$ has the dimension of mass and is chosen so that $\delta\eta_{\mathrm{infrared}}$ is a good approximation to the phase shift generated by the electromagnetic effects which take place outside of the strong interaction region, i.e., by Coulomb scattering and bremsstrahlung. One then calculates the input for (4) with a finite photon mass λ, does the integration, drops the part which diverges like $\ln(\lambda/|g(M)|)$, and takes the limit $\lambda \to 0$. The calculated value of δM will no longer depend on λ but may depend on the choice of g. This circumstance was discussed at length in (I). The relevant points were (i) if one keeps only a few nearby cuts in (4), the best estimate for δM will be obtained by choosing $g(W)$ in the specific manner described above; (ii) if one systematically improves his estimate by keeping more and more distant singularities in (4), the result will become independent of the choice of $g(W)$.

The form of $g(W)$ due to Coulomb scattering alone is obtained as follows. The only $T=\frac{1}{2}$ state undergoing Coulomb scattering is the $\pi^- p$ component of the $T_3 = -\frac{1}{2}$ state which enters with a Clebsch-Gordan coefficient $\sqrt{2}/\sqrt{3}$. Using $\delta\eta = \eta(+\frac{1}{2}) - \eta(-\frac{1}{2})$, we find that $\delta\eta_{\mathrm{infrared}}$ equals minus two-thirds the Coulomb phase shift for $\pi^- p$ scattering in the $P_{1/2}$ state. The latter can be obtained from the photon exchange diagram, and one finds for $|W-M| \ll M$ (here we are interested only in the region around the nucleon pole)

$$\delta\eta_{\mathrm{infrared}} \approx \tfrac{2}{3}\alpha[(W-M)/q]\ln(\lambda e/2q) + 0(\lambda), \quad (5)$$

where $e = 2.718\cdots$. From Eq. (5) we extract $g(W) \approx 2q/e$. At the nucleon pole, $|q|$ is approximately equal to μ, and whenever a $\ln\lambda$ divergence appears in our calculations, we will drop the part which diverges like $\ln(\lambda e/2\mu)$. Since the photon exchange diagram is gauge invariant, this method for removing spurious divergences is gauge invariant.

In addition to the Coulomb terms, there are also infrared divergent "bremsstrahlung diagrams" in which a photon connects initial and final charge lines. We now give reasons why the bremsstrahlung terms can be neglected in our particular problem. In a low-energy collision, the bremsstrahlung can be computed by finding the energy which would be radiated by the classical currents of the incoming and outgoing particles. According to the last paragraph of Sec. II, we need only consider the interference terms between the isovector and isoscalar parts of the currents. The isoscalar current comes only from the isoscalar magnetic moment of the nucleon and the isoscalar part of the nucleon recoil current. The isoscalar moment is small, and at low energies nucleon recoil is of order q/M. Furthermore, the "bremsstrahlung diagrams" are always of order α/π so that, near the pole, the bremsstrahlung contribution to $\delta\eta_{\mathrm{infrared}}$ will be of order $(q/M)(\alpha/\pi)$ and can be safely neglected.

IV. APPROXIMATE EVALUATION OF THE DISPERSION RELATION

In (I) it was indicated that if the infrared divergent terms are removed as outlined above, there is reason to believe that (4) is dominated by low-mass singularities. We will now estimate δM by keeping only those singularities. We will now estimate δM by keeping only those singularities which lie roughly in the region $|W-M| \lesssim 4\mu$. The calculation is organized as follows: First, we estimate the contribution of the kinematic corrections to the unperturbed strong interaction singularities. We next compute the effect of electromagnetic corrections to the nucleon exchange cut and then calculate the contribution of the photon exchange cut. It turns out that these two cuts plus the kinematic corrections yield a value of δM, which is in good agreement with experiment. Finally, we estimate the effect of the other nearby singularities and fortunately find that they have little effect on δM.

In order to carry out the above program, we need an expression for the unperturbed denominator function D. To a first approximation, we can set

$$D(W) \approx W - M. \quad (6)$$

A somewhat more sophisticated approximation which is better behaved at large W can be obtained by making a one-pole approximation to the cut in D, which gives

$$D(W)\approx(W-M)[(W_0-M)/(W_0-W)]. \qquad (7)$$

We can fix W_0 by comparing (7) with the denominator function derived by Balázs.[10] Setting $(W_0-M)=9\mu$ in (7) yields an expression which approximates Balázs' result to within a few percent throughout the range of interest.[11]

We now proceed with the calculation of δM.

A. Singularities of Kinematic Origin

Since the pion mass differences transform like T_3^2, we need consider only the kinematic effects of the nucleon mass difference. In the energy range under consideration, nucleon recoil can be neglected,[12] and in this approximation W and M enter into the kinematics only in the combination $(W-M)$. It is not difficult to convince oneself that this implies that the net effect of the kinematically induced singularities must be to shift the mass of a proton or neutron by an amount equal to the mass shift of its constituent nucleons. Since the proton is two-thirds π^+n and one-third π^0p, while the neutron is two-thirds π^-p and one-third π^0n, the total contribution of the singularities of kinematic origin must be $\approx-\delta M/3$. This is clearly a self-consistency requirement which arises because we have considered nucleons as bound states of nucleons and pions.

B. Corrections to Nucleon Exchange

Nucleon exchange in the u channel gives rise to a short cut which, for practical purposes, can be considered as a pole at $W=M$. Electromagnetic corrections to nucleon exchange will come from changes in the π-N coupling constants and the masses of the exchanged nucleons. Since $D(M)=0$, it follows from (4) that a change in the residue of the pole will not affect δM. The changes in the position of the pole due to the mass shifts of the exchanged nucleons can be easily calculated. Near the pole $D\approx W-M$ and using (4), one finds that electromagnetic corrections to the crossed nucleon pole contribute $+5\ \delta M/27$ to the mass difference. Again, we have a self-consistency requirement. The net contribution of the kinematic effects and crossed nucleon pole is $-4\delta M/27$. There are no other low-lying singularities which are proportional to δM.

One should note that we are not trying to make the mass difference "bootstrap" itself. In order to obtain a nonvanishing mass difference, we must introduce a driving force, i.e., electromagnetism. The reaction of the mass difference back on itself is only about a 15% effect.

C. Photon Exchange

Photon exchange gives rise to the most important singularity. This cut is contained in the photon exchange amplitude

$$\delta A_\gamma=\frac{\alpha}{6M^2}\left\{\frac{(W+M)^2-\mu^2}{(W-M)^2-\mu^2}(W-M)I_1+(W+M)I_2\right\},$$

$$I_1=\int_{-1}^{1}\frac{x}{t-\lambda^2}F_\pi(t)F_{1s}(t)dx, \qquad (8)$$

$$I_2=\int\frac{1}{t-\lambda^2}F_\pi(t)F_{1s}(t)dx,$$

$$t=-2q^2(1-x),$$

where F_π is the pion form factor, F_{1s} is the nucleon isoscalar charge form factor, and we have neglected the small isoscalar anomalous magnetic moment of the nucleon. Since the form factors have been included in its definition, δA_γ includes not only the singularity due to the photon intermediate state, but also a number of other t-channel processes. For example, we have included the singularity arising from $N\bar{N}\to\omega\to2\pi$ with the approximation $\omega\to\gamma\to2\pi$ for the $\omega\to2\pi$ amplitude.

With the choice $m_\rho^2/(m_\rho^2-t)$, $m_\rho\approx750$ MeV for F_π, one can verify that Eq. (4) will be sensitive only to the low-t behavior of F_{1s}, and a sufficiently general expression for the latter is $F_{1s}=1-c+cm_s^2/(m_s^2-t)$, where m_s is some effective resonance mass. The best fit to the low-t behavior[13] of F_{1s} is obtained with $c\approx1$ and $m_s^2\approx20\mu^2$.

Upon inserting the above form factors into (8), one finds that all the important singularities of the functions I_1 and I_2 lie in the region[14] $|W-M|\leq m_\rho/2$. In this region one of our approximate expressions (6)–(7) for D should be adequate. Using the straight-line approximation (6) for D and substituting (8) into (4), one finds that the photon exchange contribution to δM is approximately[14,15]

[10] L. Balázs, Phys. Rev. **128**, 1935 (1962).

[11] Physically, $(W_0-M)^{-1}$ should correspond roughly to the range of the forces which bind the nucleon. Since N^* exchange is generally believed to be the most important of these forces, setting $W_0-M=9\mu\approx M_{N^*}$ would seem to be very reasonable. It might be objected that N^* exchange also provides a longer range force at low energies; however, the short-range part is more important in determining the behavior of D.

[12] G. Chew, M. Goldberger, F. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

[13] C. de Vries, R. Herman, and R. Hofstadter, Phys. Rev. Letters **9**, 381 (1962).

[14] In addition to P-wave cuts near $W=M$, δA_γ has S-wave cuts (Refs. 8 and 9) near $W=-M$. The latter are outside of our region of interest and will be neglected. One can verify that setting $W+M=2M$ and $q^2=(W-M)^2-\mu^2$ in the part of δA_γ containing I_2 will have little effect on the contribution to δM of the P-wave cuts and will suffice to make the S-wave cuts negligible.

[15] The integrations leading to (9) and (11) can be carried out as follows: By explicit calculation, one finds that the cuts in δA_γ extend a finite distance into the left half-plane. The contour L in (4) will therefore be a closed loop around these cuts. With the approximation (6) for D, the integrand has no further singularities in the W plane. Expanding the contour to infinity yields (9). If one uses (7) for D, the integrand goes as $1/W^2$ at infinity, but has a pole at $W=W_0$. A simple contour integration gives (11).

$$-\frac{5}{9}\frac{\alpha}{f^2}\frac{\mu^2}{M}\left[\ln(m_\rho/\lambda)-c\frac{m_\rho^2}{m_\rho^2-m_s^2}\ln(m_\rho/m_s)\right]. \quad (9)$$

According to our earlier discussion, the spurious infrared divergence is to be removed by dropping the part proportional to $\ln(e\lambda/2\mu)$, which yields

$$-\frac{5}{9}\frac{\alpha}{f^2}\frac{\mu^2}{M}\left[\ln\left(\frac{em_\rho}{2\mu}\right)-c\frac{m_\rho^2}{m_\rho^2-m_s^2}\ln(m_\rho/m_s)\right]. \quad (10)$$

For $m_s^2=20\mu^2$ and $c=1$, Eq. (10) is numerically equal to -1.4 MeV.

The cuts in δA_γ are arranged such that, to zeroth order in $1/M$, the photon exchange contribution vanishes when D is approximated by a straight line. [Note that (10) contains a factor μ/M.] Because of this circumstance, the corrections to (10) due to the curvature of D are not entirely negligible. Taking the one-pole approximation (7) for D, one finds a photon exchange contribution of[14,15]

$$-\frac{\mu^2}{3f^2}(W_0-M)^2\frac{d}{dW}((W-M)\delta A_\gamma(W))_{W=W_0}. \quad (11)$$

After removing the spurious infrared divergence and setting $W_0=9\,\mu$, $c=1$, and $m_s^2=20\,\mu^2$, one finds that (11) has the numerical value -1.6 MeV. Reasonable upper and lower bounds[11] on W_0 would be $5\,\mu\lesssim W_0-M\lesssim 15\,\mu$. Variation of W_0 over this region cannot change the estimated photon exchange contribution by more than about ± 0.1 MeV.

The singularities of kinematic origin contribute $-\delta M/3$, the crossed nucleon pole $+5\,\delta M/27$ and photon exchange -1.6 MeV. Summing these contributions, we find

$$(31/27)\delta M\approx -1.6\text{ MeV}$$

or

$$\delta M\approx -1.4\text{ MeV}, \quad (12)$$

which is in remarkable agreement with the experimental value of -1.3 MeV.

In the subsection D, we will show that the remaining low mass singularities are very weak so that (12) emerges as our final estimate for δM. One should note that (i) since the photon exchange amplitude is gauge invariant, (12) is gauge invariant. (ii) The photon exchange contribution given by (10) diverges if *both* m_s and m_ρ tend to infinity. This is due to the bad asymptotic behavior of the straight-line denominator function (6). With a better behaved D such as (7), the photon exchange contribution is finite without form factors. For this reason, (11) and (12) are not particularly sensitive to the detailed behavior of the form factors. The form factors should, of course, be included and are necessary to obtain the observed value for δM.

D. Other Low Mass Singularities

We will now make a survey of the remaining nearby singularities and will find that they are very weak.

(a) t-Channel Cuts

In this channel we have the process $N\bar{N}\rightarrow 2\pi$. We will first consider electromagnetic corrections to ρ exchange. Since the ρ-mass differences transform like T_3^2, they will not affect δM. Furthermore, one can convince himself that electromagnetic corrections to the ρ-π coupling constants must also transform like I or T_3^2 and need not be considered. This leaves only corrections to the ρ-N coupling constants. Since the ρ and ω masses are nearly equal, the process $\rho^0\rightarrow\gamma\rightarrow\omega\rightarrow N\bar{N}$ could produce unusually large electromagnetic effects at the $\rho N\bar{N}$ vertex. However, by including the form factors in the photon exchange amplitude, we have already taken these particular corrections into account. The effect of further corrections to the ρN couplings can be estimated as follows. It can be shown that, among the possible splittings of the ρN coupling constants,[16] only a difference in the magnitudes of $f_{\rho^0 nn}$ and $f_{\rho^0 pp}$ can affect δM. We define $\Delta=|(|f_{\rho^0 pp}|-|f_{\rho^0 nn}|)/f_{\rho^0 pp}|$ and assume that Δ is about one percent. One can then insert the ρ-exchange amplitude into (4) and estimate the effect on δM. Keeping the nearby part of the ρ cut would, for $\Delta\approx 1\%$, change the calculated value of δM by less than ± 0.05 MeV.

We must also consider the ϕ and ω intermediate states. A number of authors[17–19] have pointed out that mechanisms like $\omega\rightarrow\gamma\rightarrow 2\pi$ and $\phi\rightarrow\gamma\rightarrow 2\pi$ could lead to anomalously large amplitudes for the electromagnetic transitions $\omega\rightarrow 2\pi$ and $\phi\rightarrow 2\pi$. One will recall, however, that by including the form factors in the photon exchange amplitude we have already taken the $\omega(\phi)\rightarrow\gamma\rightarrow 2\pi$ mechanism into account. It is not possible to estimate the remaining part of the $\omega(\phi)\rightarrow 2\pi$ amplitude but there does not appear to be any reason to believe that it is particularly large. An additional $\omega\pi\pi$ coupling[16] of order $f_{\omega\pi\pi}^2/4\pi\approx\alpha$ would change our estimate for δM by only a few percent.

To round out the survey of low mass singularities in the t channel, we consider the $\pi+\gamma$ intermediate state. The effect of this cut can be estimated as follows. To contribute to the mass difference, the photon must connect one isoscalar and one isovector vertex. We expect that $\pi+\gamma\rightarrow 2\pi$ is dominated by $\pi+\gamma\rightarrow\rho\rightarrow 2\pi$ which requires an isoscalar photon. On the other side of the diagram, the amplitude for $N\bar{N}\rightarrow\pi+$ (isovector γ) is probably dominated by $N\bar{N}\rightarrow\omega\rightarrow\pi+\gamma$ (the amplitude for $\phi\rightarrow\pi+\gamma$ is expected to be very small[19]). Putting the diagram together, we have $N\bar{N}\rightarrow\omega\rightarrow\pi+\gamma\rightarrow\rho\rightarrow 2\pi$, which resembles the one-photon exchange diagram with the photon replaced by $\pi+\gamma$. The relative importance of these two diagrams can be deduced by comparing their contributions to the

[16] The vector meson coupling constants $f_{\rho NN}$ and $f_{\omega\pi\pi}$ are defined in analogy with the electric charge e. We take $f_{\rho NN}^2/4\pi\approx\frac{1}{2}$.
[17] Y. Nambu and J. J. Sakurai, Phys. Rev. Letters **8**, 79 (1962).
[18] M. Gell-Mann, D. Sharp, and W. Wagner, Phys. Rev. Letters **8**, 261 (1962).
[19] R. Dashen and D. Sharp, Phys. Rev. **133**, B1585 (1964).

imaginary part of the off-mass-shell amplitude for $\omega \to \rho$. With the usual definition for the coupling constants,[18,19] one finds that the single-photon intermediate state produces an imaginary part given by

$$\mathrm{Im}_\gamma T(\rho \to \omega;\, x) = \pi\delta(x)\gamma_{\rho\gamma}\gamma_{\omega\gamma}/\mu^2, \quad (13)$$

while the $\pi+\gamma$ intermediate state yields an imaginary part of

$$\mathrm{Im}_{\pi+\gamma} T(\rho \to \omega;\, x) = (f_{\omega\pi\gamma}f_{\rho\pi\gamma}/4\pi)(\mu^4/24)(x-1)^3/x, \quad (14)$$

where we have introduced the dimensionless mass variable $x=t/\mu^2$. The product $f_{\omega\pi\gamma}f_{\rho\pi\gamma}/4\pi$ is expected[19] to be on the order of $(0.5)(\gamma_{\rho\gamma}\gamma_{\omega\gamma})/(m_\rho^2 m_\omega^2\mu^2)^{-1}$, which means that (14) is roughly equal to

$$\mathrm{Im}_{\pi+\gamma} T(\rho \to \omega;\, x) \approx (3\times 10^{-5})(\gamma_{\rho\gamma}\gamma_{\omega\gamma}/\mu^2)(x-1)^3/x. \quad (15)$$

Comparing (13) and (15), it is easy to see that the nearby part of the $\pi+\gamma$ cut will have a negligible effect on δM.

(b) s-Channel Cuts

Here we have to consider the inelastic processes $N+\pi \to N+\gamma \to N+\pi$ and $N+\pi \to N+\pi+\gamma \to N+\pi$. It follows from the last paragraph of Sec. III that the $N+\pi+\gamma$ (bremsstrahlung) cut will be negligible. The following physical argument will show that the $N+\gamma$ cut is also weak. At low energies the π-N system can radiate a *real* photon through the spin-flip current of the nucleon, the recoil current of the pion, or the formation and radiative decay of the (3–3) resonance. The latter two-photon vertices are by far the most important, but both are pure isovector. The only available isoscalar vertex comes from the nucleon spin-flip current and has a strength $(\alpha)^{1/2}(\mu_p+\mu_n)\, k/2M$, where the μ_p and μ_n are the nucleon magnetic moments and k is the photon momentum. To estimate the strength of the $N+\gamma$ cut, we multiply $\alpha^{1/2}(\mu_p+\mu_n)k/2M$ by the strength of the isovector vertices$\approx\alpha^{1/2}$ and the strength squared of the πN interaction in the $J=\frac{1}{2}^+$, $T=\frac{1}{2}$ state. At low energies, the latter is on the order of $f^2\approx 0.08$. For $k=\mu$, the discontinuity across the cut will then be on the order of $\alpha f^2(\mu_p+\mu_n)\mu/2M\approx 0.005$ g, which is down by a factor of 1/200 as compared to the discontinuity across the photon exchange cut. A straightforward calculation based on the photoproduction amplitudes of Chew, Goldberger, Low, and Nambu[20] confirms the above estimate. The nearby part of the $N+\gamma$ cut could have at most a 2% effect on δM.

(c) u-Channel Cuts

This channel also involves the process $N+\pi \to N+\pi$, and one can use the static crossing relations[12] to estimate the nearby part of the u-channel cuts; one finds, in the usual notation,[12]

$$\begin{aligned}\mathrm{Im}\delta A(M-W) &= \tfrac{1}{9}[\mathrm{Im}\delta\eta_{11}(\cos 2\eta_{11}/q^3) - 4\,\mathrm{Im}\delta\eta_{31}(\cos 2\eta_{31}/q^3) \\ &\quad - 4\,\mathrm{Im}\delta\eta_{13}(\cos 2\eta_{13}/q^3) + 16\,\mathrm{Im}\delta\eta_{33}(\cos 2\eta_{33}/q^3)] \\ &\quad + \tfrac{1}{9}[\mathrm{Re}\delta\eta_{11}(\sin 2\eta_{11}/q^3) - 4\,\mathrm{Re}\delta\eta_{31}(\sin 2\eta_{31}/q^3) \\ &\quad - 4\,\mathrm{Re}\delta\eta_{13}(\sin 2\eta_{13}/q^3) + 16\,\mathrm{Re}\delta\eta_{33}(\sin 2\eta_{33}/q^3)]. \end{aligned} \quad (16)$$

The first term in square brackets comes from the inelastic intermediate states $N+\pi+\gamma$ and $N+\gamma$. The $N+\pi+\gamma$ cut will be weak for the same reason as in the s channel. Turning to the $N+\gamma$ cut, we note that the sequence $N+\pi \to N+\gamma \to N+\pi$ with both the initial and final pions and nucleons in $T=\frac{3}{2}$ states requires two isovector photon vertices and cannot affect the mass difference. The $N+\gamma$ intermediate state therefore contributes only to $\mathrm{Im}\delta\eta_{11}$ and $\mathrm{Im}\delta\eta_{31}$. Since the strength of the low-energy π-N interaction in the (3,1) state is of the same order as in the (1,1) state, our estimate for the s-channel $N+\gamma$ cut also holds for its u-channel counterpart, and we see the latter will have little effect on δM.

The second term in square brackets on the right-hand side of (16) comes from electromagnetic corrections to elastic πN scattering. The largest part of the $\mathrm{Re}\delta\eta$'s should come from Coulomb scattering. One can replace the $\mathrm{Re}\delta\eta$'s in (16) by the Born approximation to the Coulomb phase shift, remove the infrared divergence by dropping the part which diverges like $\ln(e\lambda/2\mu)$ and use the observed strong interaction η's to estimate the effect of this cut. Since $\sin 2\eta_{33}$ changes sign at resonance, the (3,3) resonance term which usually dominates the left cut in pion-nucleon processes has very little effect on δM. The remaining phase shifts η_{11}, η_{31}, and η_{13} are very small, and one finds that the crossed πN cut probably has less than a 4 or 5% effect on the mass difference.

To summarize this section: Photon exchange and corrections to nucleon exchange and the kinematics lead to an estimate of -1.4 MeV for δM. The net contribution of all the other cuts which lie in the region $|W-M|\lesssim 4\mu$ is almost certainly less than ± 0.1 MeV.

V. CONCLUSIONS

In (I) it was argued that the high mass singularities, which we have neglected, will contribute mostly to an infrared divergent term which cancels the spurious divergence encountered above. During our treatment of potential scattering, we worked out an example in which the "strong" potential was short ranged, and the "electromagnetic" potential was cut off by form factors at small distances. There, we were able to remove the infrared terms from the dispersion integral and show explicitly that the remaining integral was completely dominated by the nearby singularities. Because of the similarity between the potential theory example in (I)

[20] G. Chew, M. Goldberger, F. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

and our present calculation of δM, and because the physics which underlies the infrared divergence is the same in both cases, one can be fairly confident that the above conjecture is correct. Assuming this to be true, none of the neglected singularities in δA is likely to have a large effect on δM, and the agreement between the theoretical and experimental values of the mass difference is not accidental.[21]

In conclusion, we discuss what physical interpretation may be placed on the calculation. We have found that the dispersion integrals are dominated by the photon exchange term connecting the isovector $\pi\pi\gamma$ vertex to the isoscalar $NN\gamma$ vertex. The isoscalar anomalous magnetic moment is small, so the $NN\gamma$ vertex essentially reduces to the isoscalar Dirac term. For the purposes of the present argument, it is convenient to add in the photon exchange connecting the isovector $\pi\pi\gamma$ vertex and the *isovector* $NN\gamma$ Dirac vertex which, we recall, shifts both proton and neutron masses in the same way and thus does not affect their splitting. We now have the full Dirac vertex with charge $+e$ for the proton and 0 for the neutron. In terms of this vertex, the only component of n or p exhibiting an important one-photon exchange is $\pi^- p$, which makes up two-thirds of the neutron.

Evidently the Coulomb part of the interaction is attractive for $\pi^- p$. Thus, one might expect the neutron to become *lighter* than the proton—exactly opposite to our result, not to mention experiment!

To see what is going on here, it is sufficient to take (10) as the contribution of the photon exchange cut. This formula, which was obtained with the approximation $D=(W-M)$, is much simpler than (11) and contains the essential physics of the situation. One will note that (10) contains a factor $1/M$, and it is not difficult to convince oneself that the only part of the photon exchange force between a π^- and p which can make a contribution of order $1/M$ is the interaction of the Dirac $(e/2M)$ magnetic moment of the proton with the magnetic field produced by the moving pion. (The ordinary electrostatic attraction does not vanish as $M\rightarrow\infty$ and nucleon recoil effects will be of order $1/M^2$.) So the main effect of the $\pi^- p$ interaction is magnetic and the Coulomb term, which was expected to make the neutron lighter, does not in fact affect the neutron mass at all in the approximation D $(W-M)$!

A heuristic explanation of this result is as follows: Loosely speaking, we have considered the neutron to be made up of a π^- bound to a fixed proton. By virtue of our assumption that D is practically a straight line, we have also assumed that the forces which bind the π^- have a range which is short compared to the inverse binding energy $1/\mu$. This means that most of the time the pion will be found outside of the region in which the binding forces operate. In this outside region the pion, with binding energy μ, has zero total energy. Now the standard expressions for the charge and current densities of a spin-zero (Klein-Gordan) particle in a potential-free region are

$$\rho=(i/2\mu)(\phi^*(\partial\phi/\partial t)-(\partial\phi^*/\partial t)\phi)\,, \tag{17}$$

$$\mathbf{j}=(1/2\mu i)(\phi^*\nabla\phi-\phi\nabla\phi^*)\,. \tag{18}$$

Zero total energy implies $\partial\phi/\partial t=0$, so the charge density vanishes, and it is not surprising that the Coulomb term vanishes. On the other hand, the pion momentum does not vanish, so there will be a current which can interact with the magnetic moment of the nucleon to produce the mass difference.

ACKNOWLEDGMENT

It is the author's pleasure to acknowledge the continued advice and encouragement given to him by Professor Steven C. Frautschi.

[21] The reader who objects to our treatment of the infrared divergence should note that, according to Eq. (9), if we had given the photon a finite mass on the order of the inverse "radius" of a nucleon $\approx 2\mu$, our estimate for δM would still be of the correct sign and order of magnitude.